THE LINE JOINING CONJUGATED CONJUGANCES

FRANCESCO SALA AND DANYLO KHILKO

Abstract. For a given triangle ABC we consider a pair of isotomically conjugated points P and Q. Let P' and Q' be the isogonal conjugances of P and Q respectively. We study properties of the line $P'Q'$ and the points where $P'Q'$ meet the sides of the triangle ABC. In particular, we generalize Theorem 1 and Theorem 3 from [1].

1. Introduction and main results

The following result was obtained in [1]:

Theorem 1. *Let ABC be an acute triangle. Its altitudes AH_1, BH_2, CH_3 intersect at the point H. Denote the midpoints of the sides AB, BC, CA by M_3, M_1, M_2. Let X_A be the foot of perpendicular from A to H_2H_3. The points X_B, X_C are defined analogously. The circumcircles of the triangles $M_1M_2X_C$, $M_1M_3X_B$, $M_2M_3X_A$ have common point which lies on the Euler line.*

The proof of this result uses the specific properties of the orthocenter and the circumcircle of a triangle. So one cannot extend this result on other configurations directly. However it is natural to ask if it is possible to generalize this fact in some way. We do it in this paper introducing the following construction.

Let ABC be an acute triangle and P, Q be a pair of isotomically conjugated points. Let P' and Q' be the isogonal conjugance of P and Q respectively (for more details about isotomic and isogonal conjugance see [4] and [3]). Let the cevian traces of P on the sides BC, CA, AB are D, E, F respectively and the circles $\odot(AEF), \odot(BDF)$ and $\odot(CDE)$ intersect the circumcircle of ABC again at the points D_0, E_0, F_0.

We generalize Theorem 1 in the following way.

Theorem A. *The circles $\odot(DE_0F_0), \odot(ED_0F_0), \odot(FD_0E_0)$ are concurrent at a point M and this point lies on a line ℓ which passes through the points P', Q'.*

It is easy to see that the generalization coincides with the above Result 1 exactly when P is Nagel point of ABC.

It occurs that the crucial role in the proof of the previous Theorem play the points K_A, K_B, K_C which are the points where $P'Q'$ meets the sides of the triangle ABC and the points X, Y, Z which are defined as follows: $X = E_0F_0 \cap BC$, $Y = F_0D_0 \cap CA$, $Z = D_0E_0 \cap AB$.

Further examination of the properties of these points resulted in a generalization of Theorem 3 from [1].

Theorem B. *The lines AK_A, YZ and E_0F_0 are concurrent.*

Also, we characterize the points where $P'Q'$ meet the sides of XYZ.

Theorem C. *The lines YZ, $P'Q'$, CE_0, BF_0 are concurrent.*

Moreover, we use this Theorem to extend Theorem A as follows. Let R and S be the complement points of P, Q with respect to ABC which means that $\overline{PG} = 2\overline{GR}$ and $\overline{QG} = 2\overline{GS}$, where G is the centroid of ABC (see [5]). Denote by R', S' the isogonal conjugates of R and S respectively.

Theorem D. *The points P', Q', R' and S' are collinear.*

All the notation which was introduced above is default notation and is used throughout the paper except Lemma 3 and Lemma 4. There we use inner notation which is independent from the default one.

2. Proofs

Firstly, we prove an auxiliary result which we are going to use twice in different parts of the paper.

First of all, let us define the point U as the intersection of BE_0 and CF_0, and analogously the points V, W. We prove a fundamental

Proposition 1. *The lines AU, BV, CW are concurrent at the point R'*

In order to prove Proposition 1 we need several Lemmas which are interesting on their own.

Let the circumcircles of BFC and BDA intesect secondly at N_B, the circumcircles of BEC and CDA intersect secondly at N_C (see Fig. 2).

Denote by P_B, P_C, P_D the points which are symmetric to P with respect to N_B, N_C, D correspondently.

The next Lemma

Lemma 1. *The circumcircle of the triangle N_CDN_B passes through the midpoint of BC and the midpoint of AP.*

Proof. Let A_1 be the midpoint of BC and K be the midpoint of AP and P_A be the symmetric point to P with respect to A_1.

It is sufficient to show that A, P_B, P_D, P_A and P_D are cyclic.

N_B is the Miquel point of the lines CF, AD, CB, BA, and the point N_C is the Miquel point of the lines BE, AD, BC, CA, hence the circumcircle of PDC passes through N_B, and the circumcircle of PBD passes through N_C. Then the triangles BN_CP are AN_CC similar, because $\angle(N_CA, AC) = \angle(N_CD, DB) = \angle(N_CP, PB), \angle(N_CC, CA) = \angle(N_CD, DP) = \angle(N_CB, BP)$. Then

$$\frac{BP}{AC} = \frac{N_CP}{AN_C} = \frac{BN_C}{N_CC}. \tag{1}$$

Obviously quadrilateral $BPCP_A$ is a parallelogram, consequently $BP = P_AC$. Also $PN_C = N_CP_C$ and $\angle(BN_C, N_CC) = \angle(BE, EC) = \angle(PN_C, N_CA)$. Using previous equalities we obtain

$$\frac{P_CN_C}{P_AC} = \frac{PN_C}{BP} = \frac{AN_C}{AC}.$$

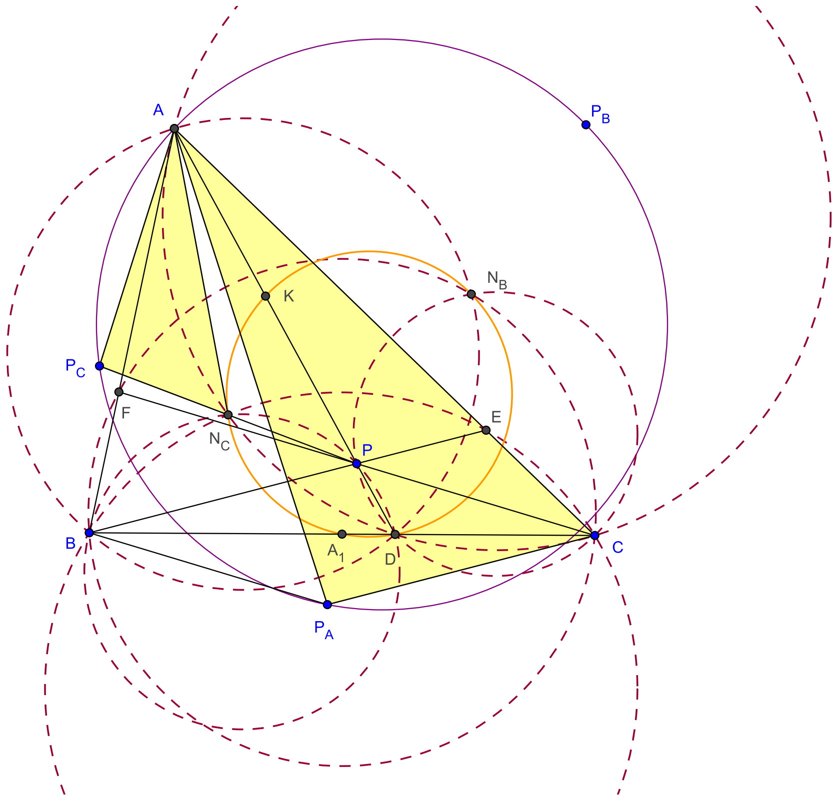

Fig. 1.

Also $\angle(P_AC, CA) = \angle(BE, EA) = \angle(BN_C, N_CC) = \angle(PN_C, N_CA) = \angle(P_CN_C, N_CA)$. Then the triangles ACP_A and AN_CP_C are similar. Hence,

$$\frac{AP_C}{AP_A} = \frac{P_CN_C}{P_AC} = \frac{AN_C}{AC}.$$

Also we have $\angle(P_CA, AN_C) = \angle(P_AA, AC)$, so $\angle(P_CA, AP_A) = \angle(N_CA, AC)$. Then the triangles P_CAP_A and N_CAC are similar, so $\angle(AP_C, P_CP_A) = \angle(AN_C, N_CC) = \angle(AD, DC)$. Analogously $\angle(AP_B, P_BP_A) = \angle(AD, DB)$. Then the quadrilateral $P_CAP_BP_A$ is inscribed because $\angle(AP_C, P_CP_A) = \angle(AD, DC) = \angle(AD, DB) = \angle(AP_B, P_BP_A)$. Now we are left to show that P_D also lies on this circle. We need that $\angle N_CDN_B = \angle P_CP_AP_B$. Using similarity of two pairs of triangles AP_CP_A, AN_CC and AP_BP_A, AN_BB we have:

$$\angle P_CP_AP_B = \angle P_CP_AA + \angle AP_AP_B = \angle N_CCA + \angle ABN_B = \angle N_CDA + \angle ADN_B = \angle N_CDN_B,$$

and we are done. $\square$

Lemma 2. *The circumcircle of the triangle N_CAN_B passes through the point D_0.*

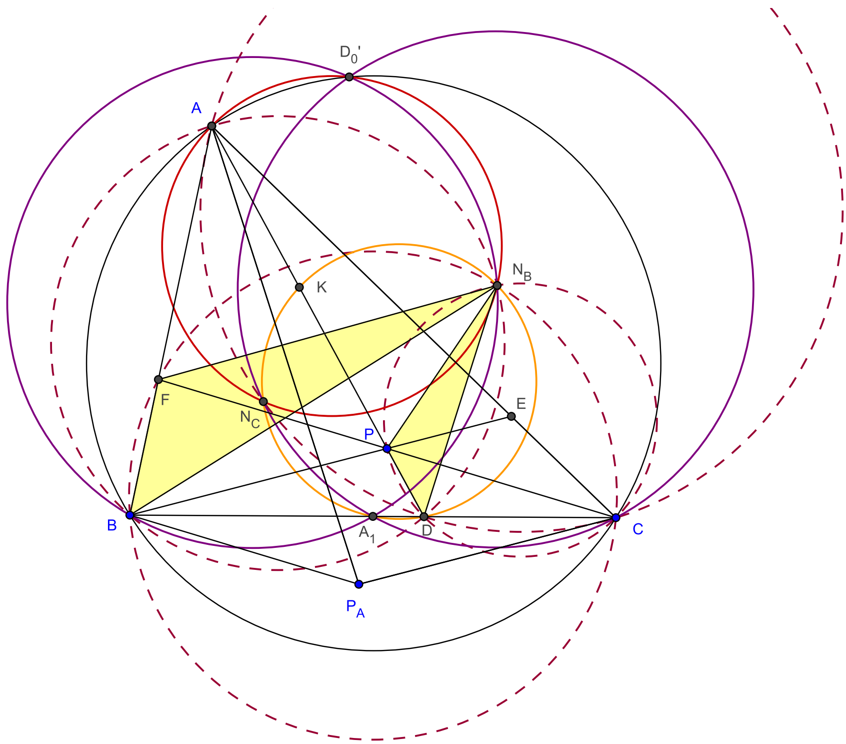

Fig. 2.

Proof. Let D_0' be the second intersection point of the circumcircles of the triangles BA_1N_B and CA_1N_C. We have $\angle(BD_0', D_0'C) = \angle(BD_0', D_0'A_1) + \angle(A_1D_0', D_0'C) = \angle(BN_B, N_BA_1) + \angle(A_1N_C, N_))$. By Lemma 1 $\angle(A_1N_C, N_CD) = \angle(A_1N_B, N_BD)$, thus $\angle(BN_B, N_BA_1) + \angle(A_1N_C, N_CC) = \angle(BN_B, N_BD) + \angle(DN_C, N_CC) = \angle(BA, AD) + \angle(DA, AC) = \angle(BA, AC)$, so D_0' lies on the circumcircle of ABC. We know that D_0 is the centre of the rotation homothety which sends BF to EC. If we prove that $\frac{BF}{EC} = \frac{D_0'B}{D_0'C}$ we will prove that $D_0' = D_0$. As A_1 is the midpoint of BC by sine law we have

$$\frac{CD_0'}{BD_0'} = \frac{\sin\angle(BD_0', D_0'A_1)}{\sin\angle(A_1D_0', D_0'C)}.$$

We have $\angle(BN_B, N_BA_1) = \angle(BD_0', D_0'A_1)$, $\angle(A_1N_C, N_CC) = \angle(A_1D_0', D_0'C)$. Then by sine law applied for the triangles A_1N_CC and A_1N_BB we have

$$\frac{\sin\angle(BN_B, N_BA_1)}{BA_1} = \frac{\sin\angle(N_BA_1, A_1B)}{BN_B},$$

and

$$\frac{\sin\angle(A_1N_C, N_CC)}{A_1C} = \frac{\sin\angle(N_CA_1, A_1C)}{N_CC}.$$

After division we obtain

$$\frac{\sin\angle(BN_B, N_BA_1)}{\sin\angle(A_1N_C, N_CC)} = \frac{\sin\angle(N_BA_1, A_1B)}{\sin\angle(N_CA_1, A_1C)} \cdot \frac{CN_C}{BN_B}.$$

By Lemma 1 the points D, A_1, N_C, N_B are cyclic. Hence, by sine law

$$\frac{\sin\angle(N_BA_1, A_1B)}{\sin\angle(N_CA_1, A_1C)} = \frac{N_BD}{DN_C}.$$

After substitution we have

$$\frac{CD_0'}{BD_0'} = \frac{N_BD}{DN_C}\cdot\frac{CN_C}{BN_B}.$$

Finally, we have two similar triangles BFN_B and DPN_B. Indeed, $\angle(PD, DN_B) = \angle(PC, CN_B) = \angle(FB, BN_B)$ and $\angle(PN_B, N_BD) = \angle(PC, CD) = \angle(FN_B, N_BB)$. Analogously, the triangles CEN_C and DPN_C. Hence,

$$\frac{DN_B}{BN_B} = \frac{PD}{BF},$$

$$\frac{CN_C}{N_CD} = \frac{CE}{PD}.$$

So,

$$\frac{CD_0'}{BD_0'} = \frac{DN_B}{N_CD}\cdot\frac{CN_C}{BN_B} = \frac{CE}{BF},$$

and we are done. □

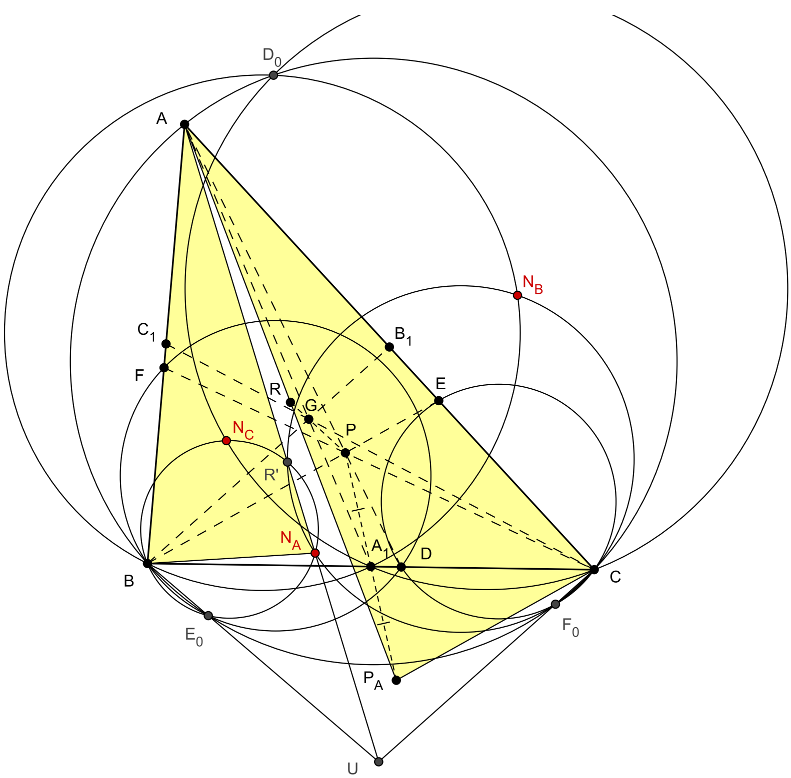

Fig. 3.

Proof of Proposition 1. We claim that the lines AN_A, AP_A are isogonal with respect to the angle $\angle BAC$ (see Fig. 3). In order to prove it just note that $\angle(BN_A, BP) = \angle(AN_A, AE)$ and $\angle(PB, PN_A) = \angle(CE, CN_A)$, so that the triangles $\triangle BN_AP, \triangle AN_AC$ are similar (analogously as it was done in the proof of Lemma 1) and $BN_A : AN_A = BP : AC = CP_A : AC$; this, with $\angle(AC, CP_A) = \angle(AE, EB) = \angle(AN_A, N_AB)$ proves that $\triangle AN_AB$ and ACP_A are directly similar, from which the claim follows. Now let G be the centroid of ABC: we observe that $PG : GR = 2$, so Menelaus theorem on the triangle PGA_1 gives us that A, R, P_A are collinear; we conclude that AP_A, BP_B, CP_C are concurrent at R, so that AN_A, BN_B, CN_C are concurrent at R'. What we are left to prove is that N_A lies on the line AU.

Now we have that

$$\angle(AN_B, AN_C) = \angle(AN_B, AD) + \angle(AD, AN_C) = \angle(BN_B, BD) + \angle(CD, CN_C),$$

where the last equivalence follows from the obvious cyclicity of AN_BDB and AN_CDC. But BN_B, CN_C intersect at R' so the last quantity equals $\angle(BR', CR')$ and the quadrilateral $AN_BR'N_C$ is cyclic. By Lemma 2 we have that D_0 belongs to this circle.

Similarly the pentagons $BE_0N_AN_CR'$ and $CF_0N_AN_BR'$ are cyclic: consequently the lines BE_0, CF_0 and $R'N_A$ concur at the radical center of these two circles and $\odot(ABC)$, so that A, N_A, U are collinear as we wanted. $\square$

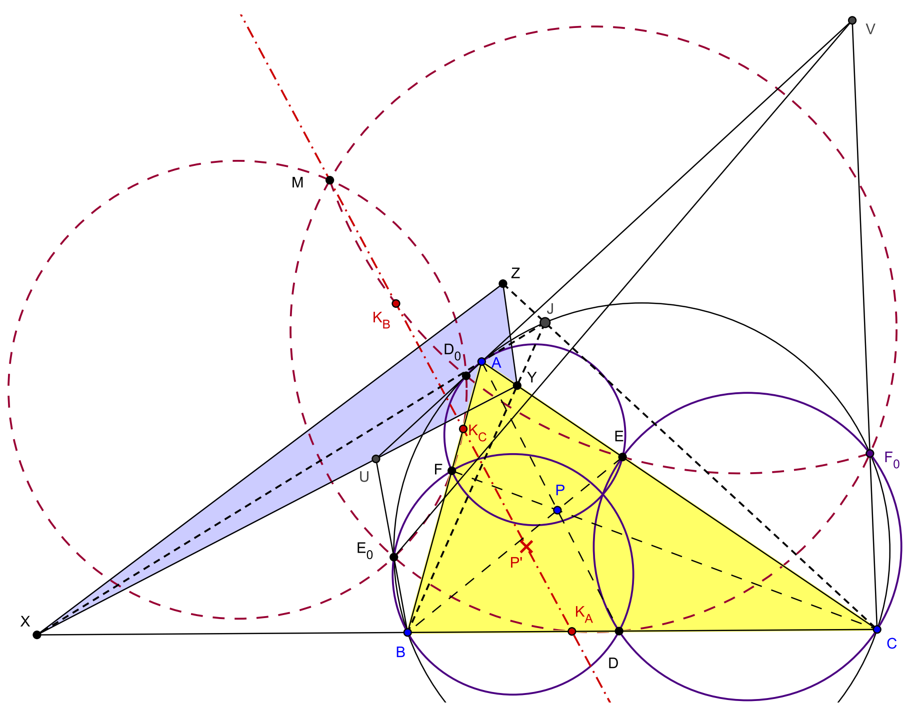

Fig. 4.

Theorem A. *The circles $\odot(DE_0F_0), \odot(ED_0F_0), \odot(FD_0E_0)$ are concurrent at a point M and this point lies on a line ℓ which passes through the points P', Q'.*

Let K_A, K_B, K_C be the second intersections of the circles $\odot(DE_0F_0)$, $\odot(ED_0F_0)$, $\odot(FD_0E_0)$ with the sides BC, CA, AB respectively (see Fig. 4). Remark that here we define these points differently from the definition which we gave in the Introduction. Later we will prove that these points lie on the line $P'Q'$ so there is no conflict in here.

Recall that X is the intersection point of the lines E_0F_0 and BC; the points Y and Z on CA and AB respectively are defined in analogous way. Now we will prove that the lines AX, BY, CY concur at a point J on $\odot(ABC)$.

In order to prove the latter claim we need a further.

Lemma 3. *Let ABC be a given triangle and P, P_1 be two cyclocevian conjugates points (that is two points such that their cevian traces D, E, F and D_1, E_1, F_1 on the sides BC, CA, AB are concylic on a circle Ω). Let us define $T_A = EF \cap E_1F_1$ and similarly T_B, T_C. Then the lines DT_A, ET_B, FT_C are concurrent at a point on Ω.*

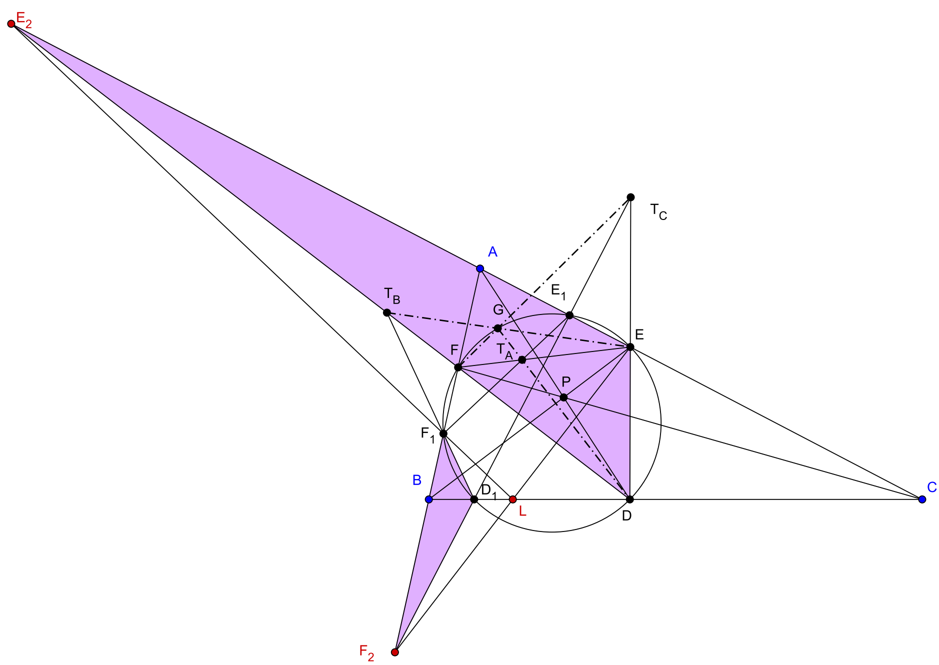

Fig. 5.

Proof. Let $E_2 = DF \cap AC$ and $F_2 = D_1E_1 \cap AB$ (see Fig. 5); finally, let $L = EF_2 \cap BC$. Clearly

$$\frac{E_2A}{E_2C} = \frac{EA}{EC}; \ \frac{F_2A}{F_2B} = \frac{F_1A}{F_1B},$$

so that by Menelaus theorem on ABC and the line F_2E

$$\frac{LB}{LC} = \frac{AE}{EC} \cdot \frac{F_2B}{F_2A} = \frac{AE}{EC} \cdot \frac{F_1B}{F_1A},$$

and E_2, F_1, L are aligned by Menelaus theorem applied on the triangle ABC. We conclude that the triangles DEE_2 and $D_1F_2F_1$ are perspective and the points $A = EE_2 \cap F_2F_1, T_B = D_1F_1 \cap DE_2$ and $T_C = DE \cap D_1F_2$ are aligned by Desargues theorem. If the line FT_C intersects Ω again at G then Pascal theorem on the hexagon $EGFF_1D_1E_1$ implies that E, G, T_B are aligned. Similarly we get that G belongs to AT_A and the Lemma is proved. □

By Proposition 1 we know that the lines AU, BV, CW concur at R'. Note that then the lines D_0U, E_0V and F_0W are concurrent and intersect at a point T which is the cyclocevian conjugate of R' with respect to UVW (see [6] for more details). Hence, comparing Fig. 5 and Fig. 4 one can detect the correspondence between the original construction and the construction from Lemma 3. Indeed, we can treat the triangle UVW as the triangle ABC from Lemma 3, then the triangle XYZ correspond to $T_AT_BT_C$ in the notion of the Lemma 3 and finally it is easy to see that the existance of the desired point J is guaranteed by the existance of the point G from Lemma 3.

Corollary 1. *The triangle XYZ is self-polar.*

Proof. The statement follows from the Brocard Theorem applied to the points A, B, C, J. □

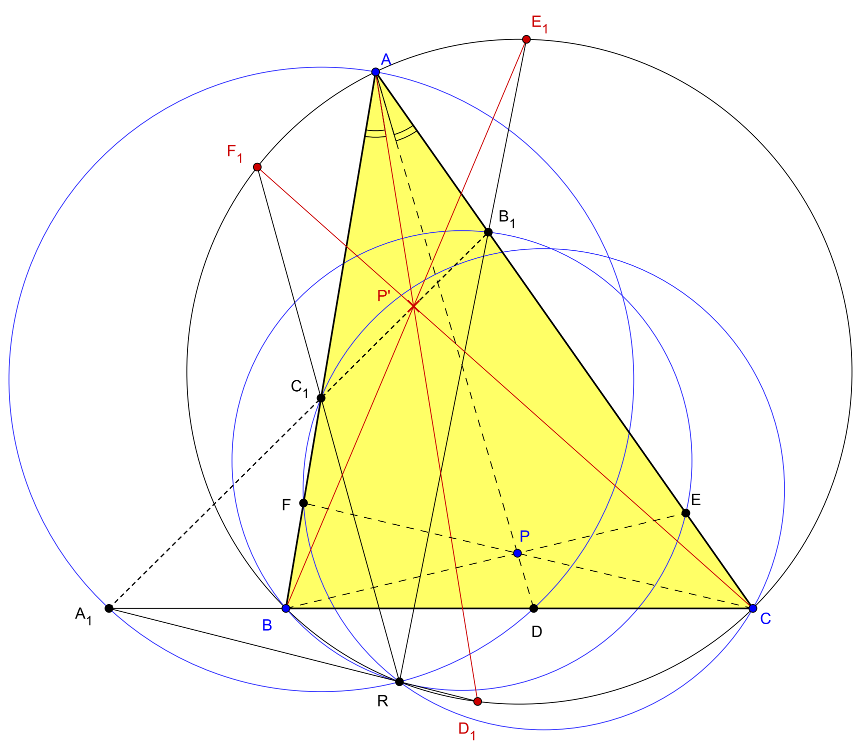

Fig. 6.

The main step of the proof is the following

Lemma 4. *Let ABC be a triangle and D, E, F be the cevian traces of a point P on the sides BC, CA, AB respectively. Let P' be the isogonal conjugate of P with respect to ABC and D_1, E_1, F_1 be the second intersection points of the* *lines AP', BP', CP' with the circumcircle of ABC. Also, let R be a point on the circle $\odot(ABC)$; moreover, let A_1 be the intersection point of the lines BC and R_1D_1. Let B_1, C_1 be similarly defined.*

Then $A_1 \in \odot(ADR)$, $B_1 \in \odot(BER)$, $C_1 \in \odot(CFR)$ and these three points are aligned on a line passing through P'.

Proof. Firstly, we will show that A_1, B_1, C_1, P' are aligned (see Fig. 6). Pascal Theorem on the hexagon $ABE_1R_1F_1C$ implies that C_1, P' and B_1 lie on a line. Similarly we have that B_1, P' and A_1 lie on a line. Now we need to prove that the point A_1 belongs to the circle $\odot(ARD)$. Two similar claims will follow immediately. We have that the triangles $\triangle ADC$ and $\triangle ABD_1$ are directly similar, because $\angle(BA, AD_1) = \angle(DA, AC)$ and $\angle(BD_1, D_1A) = \angle(DC, CA)$. Then $\angle(A_1D, DA) = \angle(CD, DA) = \angle(D_1B, BA) = \angle(D_1R, RA) = \angle(D_1R, RA_1)$. We conclude that $ADRA_1$ is a cyclic quadrilateral. $\square$

We have that $XK_A \cdot XD = XE_0 \cdot XF_0 = XA \cdot XJ$ or equivalently that K_A is the second intersection of BC with the circle $\odot(AJD)$ (see Fig. 4). Analogously, K_B is the second intersection of CA with the circle $\odot(BJD)$ and K_C is the second intersection of AB with the circle $\odot(CJD)$.

The Lemma 4 implies that K_A, K_B, K_C and P' are aligned on a line ℓ. Indeed, we can apply this Lemma to the triangle ABC and the point J on the circumcircle of ABC (see Fig. 4). Then the points K_A, K_B, K_C become the points A_1, B_1 and C_1 from the notation of Lemma 4. Remark that this fact generalizes Proposition 1 from [1].

In order to conclude that Q' lies on ℓ we just need the following.

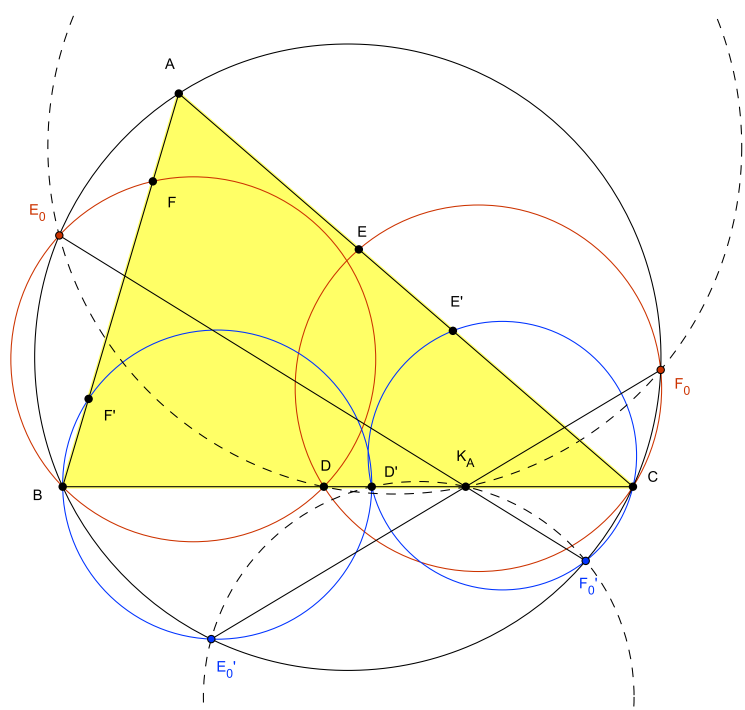

Fig. 7.

Let the points D', E', F' be the isotomic of D, E, F relatively to the respective sides (equivalently, the symmetric ones with respect to the midpoints of the sides). The circles $\odot(AE'F')$, $\odot(BD'F')$ and $\odot(CD'E')$ intersect $\odot(ABC)$ again at D'_0, E'_0, F'_0.

Lemma 5. *The second intersection point of the circles $\odot(DE_0F_0)$, $\odot(D'E'_0F'_0)$ lies on the side BC.*

Proof. Let K_A be the intersection of the lines $F_0E'_0$ and $E_0F'_0$ (see Fig. 7). Remark that we redefined the point K_A once more. Obviously, by proving that $K_A \in BC$ we will show that this new point K_A coincides with the previous one. Clearly F'_0 is the center of the similarity that sends $E'D'$ to AB, hence

$$\frac{F'_0A}{F'_0B} = \frac{F'_0E'}{F'_0D'} = \frac{AE'}{BD'} = \frac{CE}{CD},$$

we conclude that the triangles $\triangle F'_0AB, \triangle F'_0E'D', \triangle CED$ are directly similar. Then $\angle(AF'_0, AC) = \angle(AF'_0, AB) + \angle(AB, AC) = \angle(EC, ED) + \angle(AB, AC) = \angle(AB, ED)$; similarly we get $\angle(AF_0, AC) = \angle(AB, E'D')$ and $\angle(AF'_0, AF_0) = \angle(AF'_0, AC) + \angle(AC, AF_0) = \angle(AB, ED) + \angle(E'D', AB) = \angle(E'D', ED)$. Similarly, $\angle(AE'_0, AE_0) = \angle(D'F', DF)$. It immediately follows that $\angle(K_AE_0, K_AF_0) = \angle(F'_0E_0, F'_0E'_0) + \angle(E'_0F'_0, E'_0F_0) = \angle(AE_0, AE'_0) + \angle(AF'_0, AF_0) = \angle(DF, D'F') + \angle(E'D', ED) = \angle(FD, DE) + \angle(E'D', D'F')$. We can now infer that K_A belongs to the circle $\odot(DE_0F_0)$: indeed $\angle(DE_0, DF_0) = \angle(DE_0, DF) + \angle(FD, DE) + \angle(DE, DF_0) = \angle(BE_0, BF) + \angle(FD, DE) + \angle(CA, CF_0) = \angle(AE_0, AF_0) + \angle(FD, DE)$. It is known that $\angle AE_0, AF_0 = \angle D'E', D'F'$ (see, for instance, [2]). Anyway, we give the proof of it. We have two pairs of similar triangles E_0FA and E_0DC; F_0EA and F_0DB. So

$$\frac{E_0A}{E_0C} = \frac{AF}{CD} = \frac{BF'}{BD'};\ \frac{F_0A}{F_0B} = \frac{AE}{BD} = \frac{CE'}{CD'},$$

and from these equations we get similarity of the triangles AE_0C and $F'BD'$; AF_0B and $E'CD'$. Consequently, $\angle(E_0C, CA) = \angle(BD', D'F')$, $\angle(F_0B, BA) = \angle(CD', D'E')$ and $\angle(E_0A, AF_0) = \angle(E_0C, CA) + \angle(AB, BF_0) = \angle(BD', D'F') + \angle(E'D', D'C) = \angle(E'D', D'F')$. Now we just conclude that $\angle(DE_0, DF_0) = \angle(AE_0, AF_0) + \angle(FD, DE) = \angle(E'D', D'F') + \angle(FD, DE) = \angle(K_AE_0, K_AF_0)$. Similarly, K_A also lies on the circle $\odot(D'E'_0F'_0)$. All we have to do now is just to prove that $K_A \in BC$: but $K_A \in \odot(DE_0F_0)$ implies that $\angle(DK_A, DF_0) = \angle(E_0K_A, E_0F_0) = \angle(AF'_0, AF_0) = \angle(E'D', ED)$; besides, we have $\angle(DC, DF_0) = \angle(CE, EF_0) = \angle(CE, ED) + \angle(ED, EF_0) = \angle(CE, ED) + \angle(E'D', E'C) = \angle(E'D', ED) = \angle(DK_A, DF_0)$ and D, K_A, C are collinear, as we wanted. □

Using Lemma 5 we can now repeat the last arguments for the point Q instead of P to get that K_A, K_B, K_C and Q' are collinear.

Proof of Theorem A. Now let M be the second intersection point of $\odot(FE_0D_0)$ with $P'Q'$ (we have already deduced that K_C is a common point of $\odot(FE_0D_0)$ and $P'Q'$; M is an another one). Then $\angle(MK_B, MD_0) = \angle(FK_C, FD_0) = \angle(EK_B, ED_0)$ and $M \in \odot(ED_0F_0)$ (see Fig.). Analogously we get that $M \in \odot(DE_0F_0)$, as the Theorem stated.

□

The next Theorem gives generalization of Theorem 3 in [1].

Theorem B. *The lines AK_A, YZ and E_0F_0 are concurrent.*

We introduce some additional notation. Let the tangents to the circumcircle of ABC at points D_0 and F_0 intersect at the point S_B. The points S_A and S_C are defined analogously (see Fig. 2).

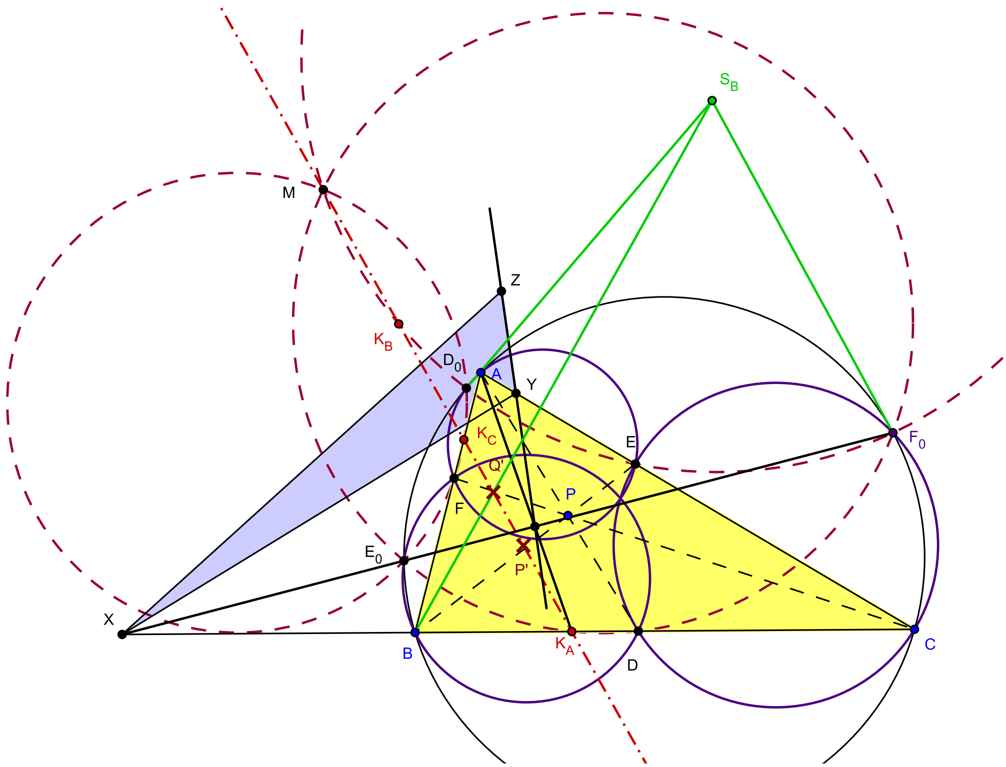

Fig. 8.

To prove Theorem B we need some preparation given below.

Lemma 6. *The lines AS_A, BS_B, CS_C intersect at the point Q'.*

Proof. It is sufficient to prove that Q' belongs to the line AS_A.

Consider the intersection point $L_A \neq D$ of the circle which passes through D and touches AB at B and the circle which also passes through D and touches AC at C (see Fig. 9). It is easy to see that $L_A \in \odot(ABC)$, Indeed, $\angle(BL_A, L_AC) = \angle(BL_A, L_AD) + \angle(DL_A, L_AC) = \angle(AB, BD) + \angle(DC, CA) = \angle(BA, AC)$. We have two pairs of similar triangles: BDL_A and ACL_A; CDL_A and ABL_A. So we have

$$(2) \qquad \frac{BD}{AC} = \frac{BL_A}{AL_A} = \frac{DL_A}{L_AC}; \; \frac{DC}{AB} = \frac{L_AC}{AL_A} = \frac{DL_A}{L_AB},$$

and

$$(3) \qquad \angle(DL_A, L_AC) = \angle(BC, CA); \; \angle(DL_A, L_AB) = \angle(CB, BA).$$

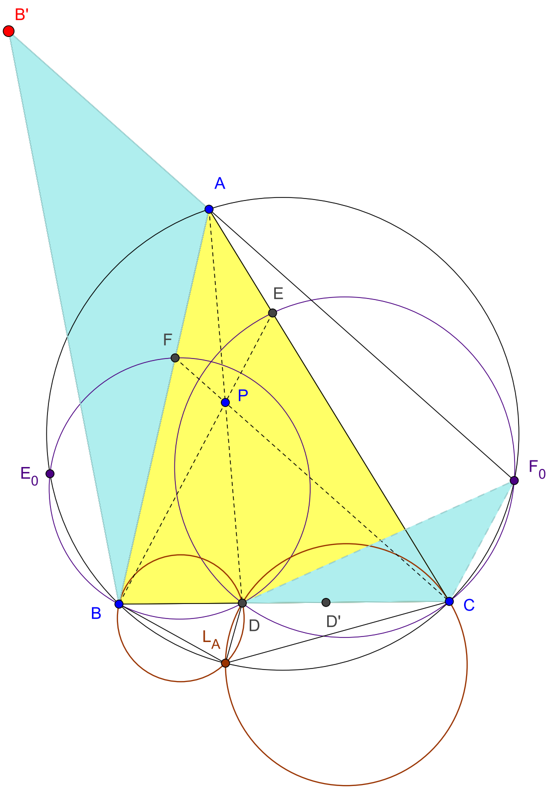

Fig. 9.

Recall that the point D' on the segment BC is such that $BD = D'C$. The triangles ABL_A and $AD'C$ are similar because $\angle(BL_A, L_AA) = \angle(D'C, CA)$ and from (2)

$$\frac{CD'}{AC} = \frac{BD}{AC} = \frac{BL_A}{AL_A}.$$

Consequently, the line AL_A passes through Q' and the problem is to prove that AL_A and tangents to $\odot(ABC)$ at E_0 and F_0 are concurrent or in other words that $BACL_A$ must be harmonic. It is well-known that the quadrilateral is harmonic iff the products of its opposite sides are equal. So we have to prove that:

$$L_AF_0 \cdot AE_0 = LE_0 \cdot AF_0,$$

or

$$\frac{AF_0}{L_AF_0} = \frac{AE_0}{L_AE_0}.$$

Let B' be a point on AF_0 for which the triangles BAB' and DCF_0 are directly similar. This point exists because $\angle(F_0A, AB) = \angle(F_0C, CB)$. We have

$$\frac{DF_0}{BB'} = \frac{DC}{AB} \tag{4}$$

and $\angle(AB, BB') = \angle(CD, DF_0)$. Since the triangles L_ADC and L_ABA are directly similar we obtain $\angle(L_AD, DF_0) = \angle(L_AB, BB')$. With the help of the equation (2) we get

$$\frac{BB'}{DF_0} = \frac{AB}{DC} = \frac{BL_A}{L_AD}, \tag{5}$$

so the triangles L_ABB' and L_ADF_0 are similar. Hence,

$$\frac{L_AF_0}{B'L_A} = \frac{DL_A}{BL_A}, \tag{6}$$

and $\angle(BL_A, L_AB') = \angle(DL_A, L_AF_0)$. Then, keeping in mind (3), $\angle(B'L_A, L_AF_0) = \angle(B'L_A, L_AD) + \angle(DL_A, L_AF_0) = \angle(BL_A, L_AD) = \angle(AL_A, L_AC)$. Obviously, $\angle(AF_0, F_0L_A) = \angle(AC, CL_A)$. Hence the triangles $B'L_AF_0$ and AL_AC are similar so

$$\frac{B'L_A}{B'F_0} = \frac{AL_A}{AC}. \tag{7}$$

By similarity of $B'AB$ and F_0CD $\angle(BB', B'A) = \angle(DF_0, F_0C) = \angle(DE, EC)$. Also $\angle(AF_0, F_0B) = \angle(AC, CB)$ so the triangles $BB'F_0$ and DEC are similar. As a consequence,

$$\frac{B'F_0}{BF_0} = \frac{EC}{DC}. \tag{8}$$

Finally, $\angle(CA, AF_0) = \angle(CB, BF_0)$, $\angle(CD, DF_0) = \angle(CE, EF_0)$, so the triangles AEF_0 and BDF_0 are similar and

$$\frac{BF_0}{AF_0} = \frac{BD}{AE}. \tag{9}$$

Now, multiply (6), (7), (8), (9):

$$\frac{L_AF_0}{AF_0} = \frac{DL_A}{BL_A} \cdot \frac{AL_A}{AC} \cdot \frac{EC}{DC} \cdot \frac{BD}{AE}.$$

Analogously,

$$\frac{L_AE_0}{E_0A} = \frac{DL_A}{CL_A} \cdot \frac{AL_A}{AB} \cdot \frac{FB}{DB} \cdot \frac{CD}{AF}.$$

We need to prove that

$$\frac{DL_A}{BL_A} \cdot \frac{AL_A}{AC} \cdot \frac{EC}{DC} \cdot \frac{BD}{AE} = \frac{DL_A}{CL_A} \cdot \frac{AL_A}{AB} \cdot \frac{FB}{DB} \cdot \frac{CD}{AF},$$

or

$$\frac{AB}{DC} \cdot \frac{BD}{AC} \cdot \frac{CL_A}{BL_A} \cdot \frac{BD}{DC} \cdot \frac{EC}{AE} \cdot \frac{AF}{FB} = 1.$$

But by Ceva Theorem applied for P and ABC we have

$$\frac{BD}{DC} \cdot \frac{EC}{AE} \cdot \frac{AF}{FB} = 1.$$

Also, by similarity of two pairs of triangles BDL_A and ACL_A; CDL_A and ABL_A (see (2))

$$\frac{AB}{DC} \cdot \frac{BD}{AC} \cdot \frac{CL_A}{BL_A} = 1,$$

and we are done. □

Recall, that J is the second intersection point of XA and the circumcircle of ABC.

Lemma 7. (i) *X, J, L_A and D are cyclic.*
(ii) *The midpoint of E_0F_0 belongs to the circle around X, J, L_A and D.*

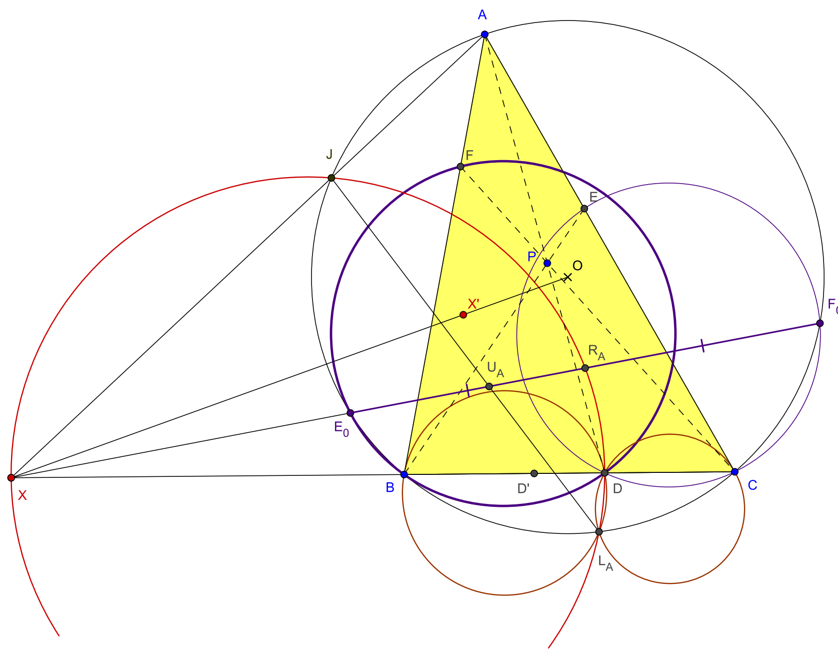

Fig. 10.

Proof. (i) It is easy, because $\angle(XD, DL_A) = \angle(CD, DL_A) = \angle(AB, BL_A) = \angle(AJ, JL_A) = \angle(XJ, JL_A)$. (See Fig. 10.)

(ii) Let $R_A \neq X$ be the second intersection point of E_0F_0 and the circumcircle of XJL_AD. Then $\angle(XR_A, R_AL_A) = \angle(XJ, JL_A) = \angle(AF_0, F_0L_A)$. Obviously, $\angle(L_AE_0, E_0F_0) = \angle(L_AA, AF_0)$. Hence, the triangles $E_0R_AL_A$ and AF_0L_A are similar, so

$$E_0R_A = AF_0 \cdot \frac{E_0L_A}{AL_A}.$$

Similarly,

$$F_0R_A = AE_0 \cdot \frac{F_0L_A}{AL_A}.$$

In the proof of Lemma 6 we have shown that the quadrilateral $E_0AF_0L_A$ is harmonic which is equivalent to

$$F_0A \cdot E_0L_A = E_0A \cdot F_0L_A.$$

Now it is easy to see that $E_0R_A = F_0R_A$. □

Lemma 8. *The lines YZ, E_0F_0 and L_AJ are concurrent.*

Proof. Let E_0F_0 and L_AJ intersect at U_A (see Fig. 10). We will show that U_A lies on the polar line of X with respect to $\odot(ABC)$. Then Corollary 1 implies the desired statement. Let O be the center of $\odot(ABC)$ and let X' be the orthogonal projection of U_A on XO. It suffices to show that X' is the image of X after inversion with respect to $\odot(ABC)$, in other words that $OX' \cdot OX = OA^2$.

By Lemma 7 we deduce that $OR_A \perp E_0F_0$, so $XX' \cdot XO = XU_A \cdot XR_A$. Also, $JU_A \cdot U_AL_A = XU_A \cdot U_AR_A$. Then

$$\begin{aligned} &OX \cdot OX' = OX^2 - XX' \cdot OX = OX^2 - XU_A \cdot XR_A = \\ &= XR_A^2 + OR_A^2 - XU_A \cdot XR_A = U_AR_A \cdot XR_A + OR_A^2 = \\ &= U_AR_A^2 + OR_A^2 + JU_A \cdot U_AL_A = OU_A^2 + JU_A \cdot U_AL_A = OA^2, \end{aligned}$$

and we are done. □

In order to prove Theorem we just have to show that U_A belongs to AK_A. Let G_A be the second intersection point of $\odot(ABC)$ and the line XL_A.

Lemma 9. *The points X, G_A, D', R_A and A are cyclic.*

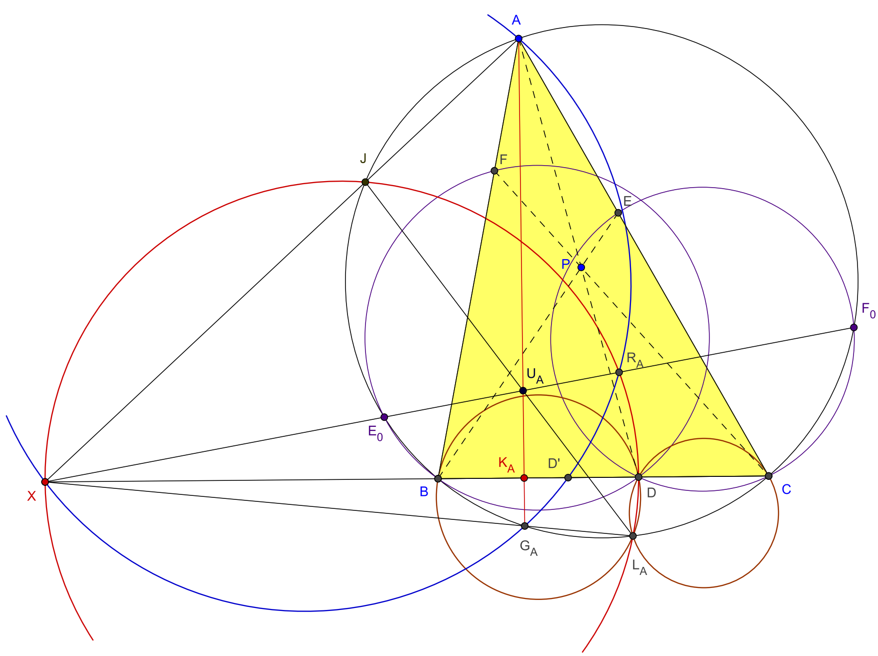

Fig. 11.

Proof. Keeping in mind similarity of the triangles ACL_A and $AD'B$ (it was proved in Lemma 6) we have $\angle(XG_A, G_AA) = \angle(L_AC, CA) = \angle(BD', D'A)$, so X, G_A, D', A lie on a circle (see Fig. 11). In the proof of Lemma 7 (ii) we have mentioned that $E_0R_AL_A$ and AF_0L_A are similar. Also, $E_0AF_0L_A$ is harmonic, hence

$$E_0R_A = F_0A \cdot \frac{E_0L_A}{AL_A} = \frac{E_0A \cdot F_0L_A}{AL_A},$$

or

$$\frac{E_0R_A}{E_0A} = \frac{F_0L_A}{AL_A}.$$

Obviously, $\angle(AE_0, E_0F_0) = \angle(AL_A, L_AF_0)$, so the triangles AE_0R_A and AF_0L_A are similar. Consequently, $\angle(E_0R_A, R_AA) = \angle(L_AF_0, F_0A) = \angle(XG_A, G_AA)$. $\square$

Proof of Theorem B. We will prove that A, K_A and G_A are collinear (see Fig. 11). If it is true then Theorem follows as AG_A, E_0F_0, JL_A are concurrent because these three lines are the radical axes of the circles $\odot(ABC)$, $\odot(XJR_ADL_A)$ and $\odot(XG_AD'R_AA)$.

We have characterized the point K_A as the second intersection of $\odot(JAD)$ with BC. Hence, $\angle(AK_A, K_AD) = \angle(AJ, JD)$. It suffices to show that $\angle(AG_A, BC) = \angle(AJ, JD)$. But using the previous results we can conclude that $\angle(AG_A, BC) = \angle(G_AA, AX) + \angle(AX, XD) = \angle(G_AL_A, L_AJ) + \angle(AX, XD) = \angle(XD, DJ) + \angle(AX, XD) = \angle(AX, JD) = \angle(AJ, JD)$, so we are done. $\square$

Theorem C. *The lines YZ, $P'Q'$, CE_0, BF_0 and G_AD_0 are concurrent.*

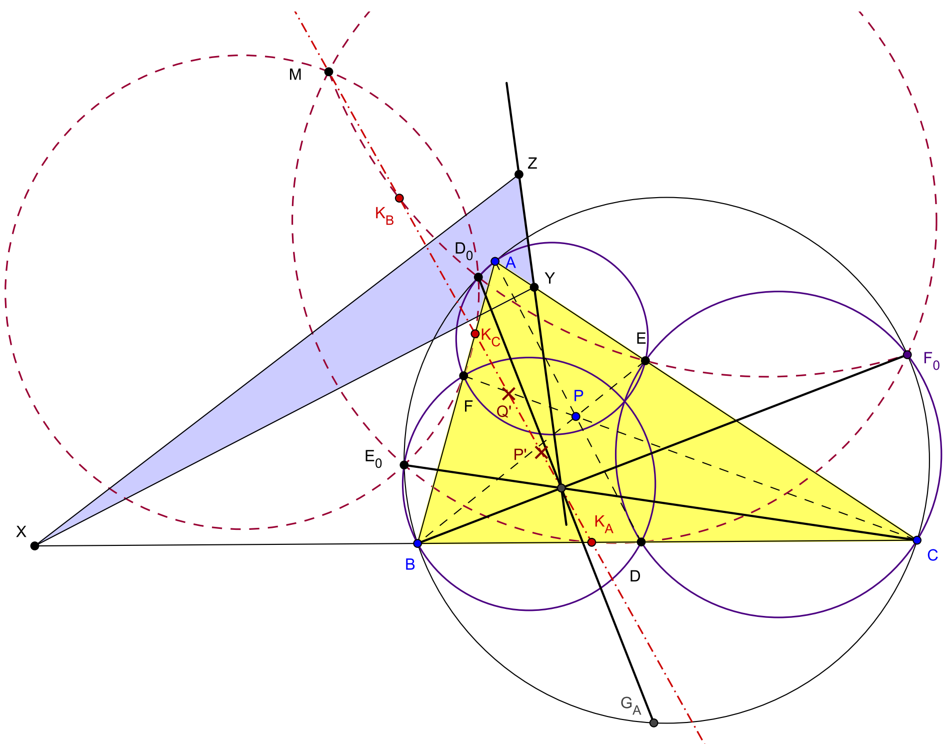

Fig. 12.

Remark that we are going to prove slightly strengthened claim than we have promised in the Introduction as now we have the point G_A defined.

To prove this Theorem we need several Lemmas.

Lemma 10. *The lines YZ, BF_0, CE_0 are concurrent.*

Proof. This is trivial. It is known that BF_0 and CE_0 intersect on the polar line of the point X which is YZ by Corollary 1. $\square$

Lemma 11. *The quadrilateral $JE_0G_AF_0$ is harmonic.*

Proof. To prove this Lemma we need to return to the previous construction (see Fig. 11). By Lemma 9 $\angle(XR_A, R_AG_A) = \angle(XA, AG_A) = \angle(JF_0, F_0G_A)$. Obviously, $\angle(R_AF_0, F_0G_A) = \angle(E_0A, AG_A)$. Then the triangles $R_AF_0G_A$ and E_0JG_A are similar. Hence,

$$\frac{E_0J}{R_AF_0} = \frac{JG_A}{G_AF_0}.$$

Analogously, $R_AE_0G_A$ and F_0JG_A are similar and

$$\frac{F_0J}{E_0R_A} = \frac{JG_A}{G_AE_0}.$$

As $E_0R_A = R_AF_0$ we conclude that

$$\frac{E_0J}{F_0J} = \frac{E_0G_A}{G_AF_0},$$

so the quadrilateral $JE_0G_AF_0$ is harmonic. $\square$

Denote the meeting point of BF_0 and CE_0 by N_A and define N_B and N_C similarly ($N_B = AF_0 \cap CD_0$, $N_C = AE_0 \cap BD_0$).

Lemma 12. *The lines CE_0, BF_0 and G_AD_0 are concurrent.*

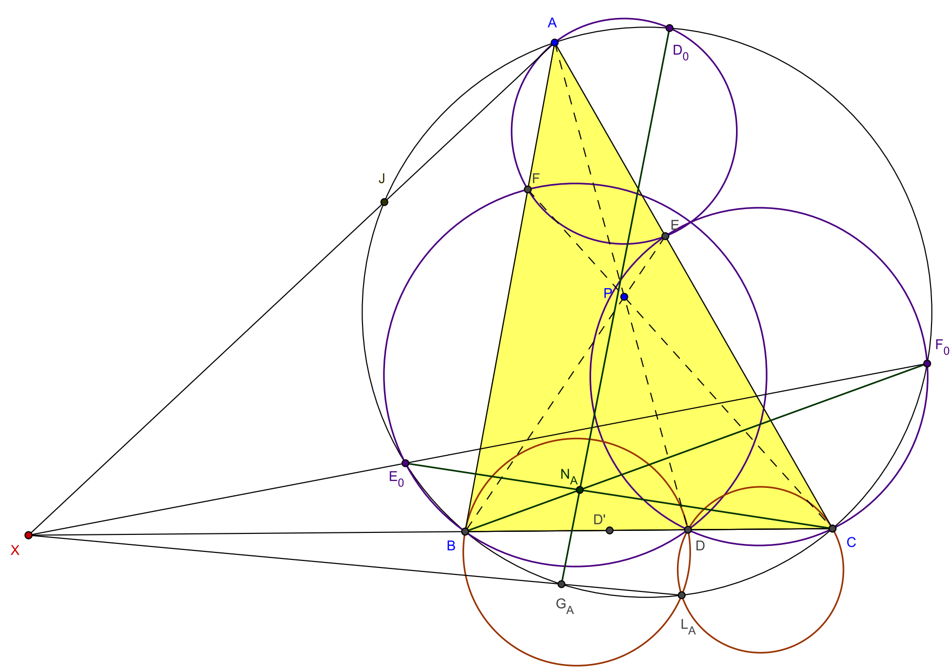

Fig. 13.

Proof. We will use the trigonometric form of Ceva Theorem for triangle $E_0D_0F_0$ and the lines E_0C, D_0G_A, F_0B (see Fig. 13). We need to prove that

$$\frac{\sin\angle E_0D_0G_A}{\sin\angle G_AD_0F_0}\cdot\frac{\sin\angle F_0E_0C}{\sin\angle CE_0D_0}\cdot\frac{\sin\angle D_0F_0B}{\sin\angle BF_0E_0}=1.$$

We can replace the ratios of sinuses by the ratios of chords or in other words we need to prove:

$$\frac{E_0G_A}{G_AF_0}\cdot\frac{F_0C}{CD_0}\cdot\frac{D_0B}{BE_0}=1. \tag{10}$$

By Lemma 11

$$\frac{E_0G_A}{F_0G_A}=\frac{E_0J}{JF_0}, \tag{11}$$

by similarity of D_0FB and D_0EC

$$\frac{D_0B}{D_0C}=\frac{BF}{CE}.$$

We will try to transform E_0J/JF_0. Obviously, we have two pairs of similar triangles: XJE_0 and XAF_0; XAE_0 and XJF_0. Hence,

$$\frac{E_0J}{AF_0}=\frac{XE_0}{XA};\quad \frac{JF_0}{AE_0}=\frac{XF_0}{XA}.$$

After dividing we get

$$\frac{E_0J}{JF_0}=\frac{AF_0}{AE_0}\cdot\frac{XE_0}{XF_0}.$$

By similarity of AEF_0 and BDF_0; CDE_0 and AFE_0:

$$\frac{AF_0}{BF_0}=\frac{AE}{BD};\quad \frac{AE_0}{E_0C}=\frac{AF}{DC}.$$

We divide equations once again and obtain

$$\frac{AF_0}{AE_0}=\frac{BF_0\cdot AE}{BD}\cdot\frac{DC}{AF\cdot E_0C},$$

hence

$$\frac{E_0J}{JF_0}=\frac{BF_0\cdot AE}{BD}\cdot\frac{CD}{AF\cdot E_0C}\cdot\frac{XE_0}{XF_0}.$$

Now we substitute these equations into the initial expression. So we need to prove that

$$\frac{BF_0\cdot AE}{BD}\cdot\frac{CD}{AF\cdot E_0C}\cdot\frac{XE_0}{XF_0}\cdot\frac{BF}{CE}\cdot\frac{F_0C}{BE_0}=1,$$

or after permutaion of factors

$$\frac{AE}{EC}\cdot\frac{CD}{DB}\cdot\frac{BF}{FA}\cdot\frac{BF_0}{E_0C}\cdot\frac{XE_0}{XF_0}\cdot\frac{F_0C}{BE_0}=1.$$

By Ceva Theorem applied for the triangle ABC and the point P

$$\frac{AE}{EC}\cdot\frac{CD}{DB}\cdot\frac{BF}{FA}=1,$$

so we need to prove that

$$\frac{BF_0}{E_0C}\cdot\frac{XE_0}{XF_0}\cdot\frac{F_0C}{BE_0}=1,$$

but this follows from the similarity of XBF_0 and XE_0C; XF_0C and XE_0B. Indeed, we have

$$\frac{BF_0}{E_0C} = \frac{XF_0}{XC},$$

and

$$\frac{F_0C}{BE_0} = \frac{XC}{XE_0},$$

so after subtitution we obtain the desired equation. □

Now we need to show that all the points N_A, N_B, N_C lie on the line $P'Q'$.

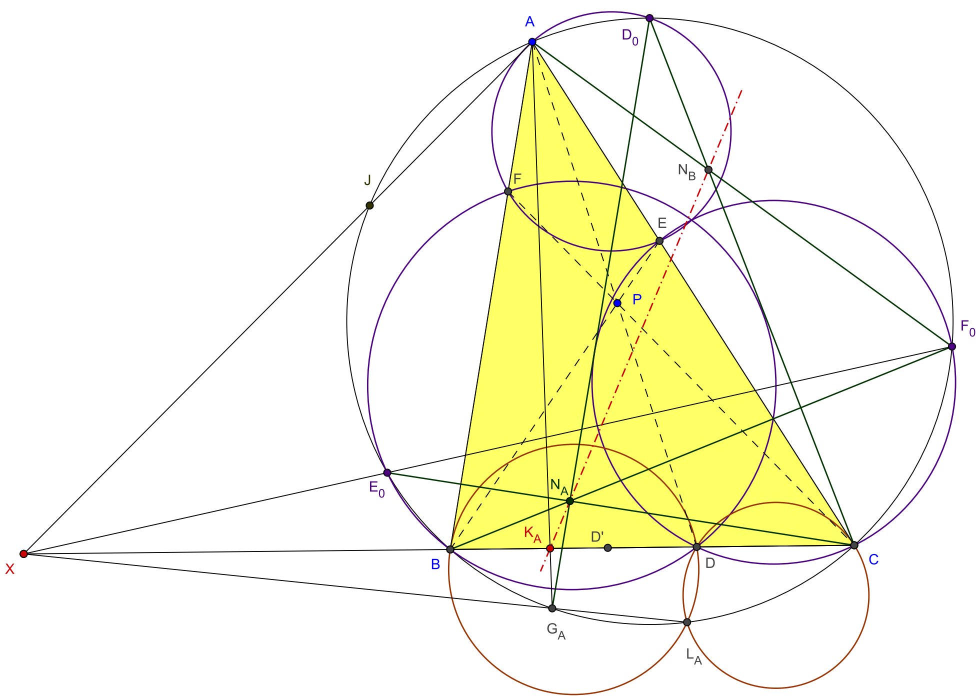

Fig. 14.

Proof of Theorem C. Apply Pascal Theorem for hexagons $AG_AD_0CBF_0$ and $AG_AD_0BCE_0$ (see Fig. 14). Using the Lemma 12 we conclude from the first one that N_A, N_B and K_A and from the second that N_A, N_C and K_A are aligned. Consequently, N_A, N_B, N_C and K_A are collinear. We can repeate the last argument to obtain that six points $N_A, N_B, N_C, K_A, K_B, K_C$ are collinear. But we know that K_A, K_B and K_C lie on the line $P'Q'$ and this finishes the proof. □

Now we will prove that R' and S' lie on $P'Q'$ or in other words the promised

Theorem D. *The points P', Q', R' and S' are collinear.*

Proof. Recall that by Proposition 1 we know that AU, BV, CW concur at R', and D_0U, E_0V, F_0W are concurrent at a point T (see Fig. 15). By Pappus' Theorem applied on the lines $\overline{VCF_0}$ and $\overline{WBE_0}$ we get that the points $R' = VB \cap WC, T = VE_0 \cap WF_0$ and $N_A = BF_0 \cap CE_0$ are collinear. Analogously we conclude that N_B ,$N_C \in R'T$. But from Theorem C we know that N_A, N_B, N_C belong to the line $P'Q'$, hence $R'T$ coincide with $P'Q'$. Finally, we can swap P and Q and then repeat the proof for S' and obtain that $S' \in P'Q'$.

We conclude that P', Q', R', S' are collinear. □

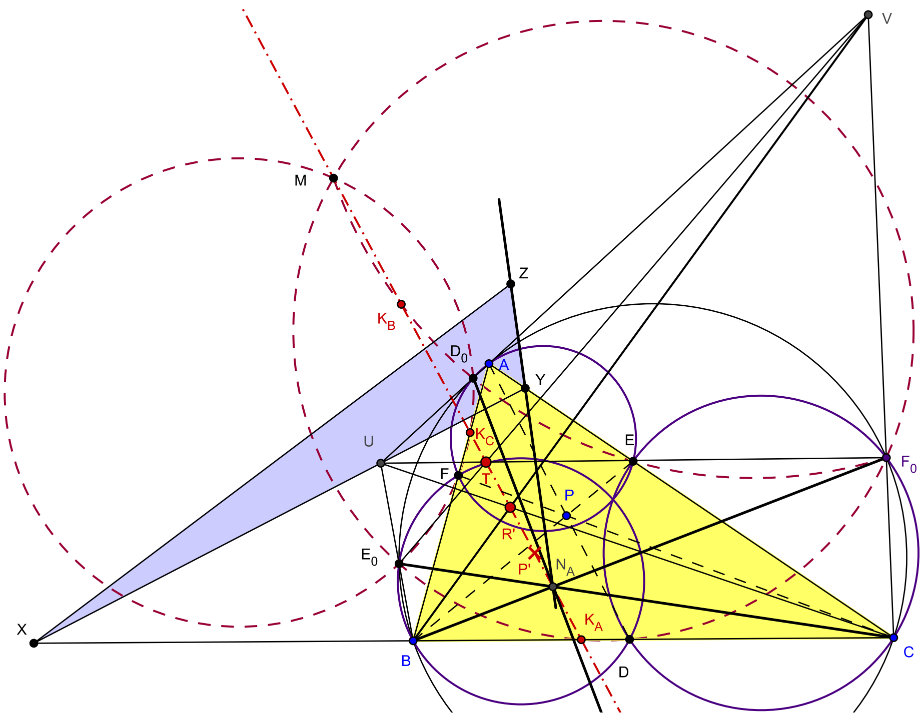

Fig. 15.

References

[1] Danylo Khilko, *Some properties of intersection points of Euler line and orthotriangle* Journal of Classical Geometry, Volume 3, 2014.

[2] Lev. A. Emelyanov and Pavel A. Kozhevnikov, *Isotomic similarity* Journal of Classical Geometry, Volume 1, 2012.

[3] Weisstein, Eric W. "Isogonal Conjugate." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/IsogonalConjugate.html

[4] Weisstein, Eric W. "Isotomic Conjugate." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/IsotomicConjugate.html

[5] Weisstein, Eric W. *Complement.* From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/Complement.html

[6] Weisstein, Eric W. *Cyclocevian Conjugate.* From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/CyclocevianConjugate.html

Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italy

E-mail address: `francesco.sala@sns.it`

Taras Shevchenko National University of Kyiv, Academician Glushkov prospectus 4-b, 03127 Kyiv, Ukaraine

E-mail address: `dkhilko@ukr.net`